3.34. Duality and Expressive Power

Here we seek to extend the concept of "dual" to bring it into contact with our discussion of expressive adequacy. Duality is extended to sets of connectives, and from there to formal languages.

1. Dual Sets. It is simple enough to extend the concept to formal languages, by way of duals of connectives.

Recall that the Connective Swap Method defined "dual of a sentence" in terms of "**dual of a connective**": the vel has its dual connective the wedge (and vice versa); and the tilde is its own dual. The idea of the dual of a connective can be extended naturally to sets of connectives. For a set of connectives, its **dual set** is just the set containing the dual of each connective in the original set. For example, the set of connectives {~, ∧} takes as it dual the set {~, ∨}.

But since we characterize a formal language as a set of connectives – referring, e.g., to the language {~, ∧}, or the language {∧, ∨} – talk of duals sets extends naturally to languages: the dual of a formal language is just its dual set. For example, the dual of language {~, ∧} is {~, ∨}. Again, the dual of formal language {~, ∧} is the language{~, ∨}.

Certain formal languages turn out to be self-duals, in the same way that certain connectives are. For instance {~} is a self-dual language, since "~" is a self-dual. But {∧, ∨} is also a self-dual language; for the dual of "∧" is "∨," and vice versa. Mapping each connective in {∧, ∨} onto its dual just yields {∧, ∨} again. In general: if each connective in the set has its dual in that same set, then that set is a self-dual.

2. Languages, Duality, and Expressive Power. We can also compare different formal languages in terms of expressive power – the truth tables picks out by the sentences of that language.

The language {~, ∧, ∨}is *expressively adequate* , since for any truth table there is some {~, ∧, ∨} sentence which picks out that truth table. (DNF sentences, for example, are all members of language {~, ∧, ∨}, and every truth table is guaranteed to have a corresponding DNF sentence.) By contrast, the language {~} is *expressively inadequate*, since there are some truth tables which aren't picked out by any {~} sentence. So we say that language {~, ∧, ∨} is **expressively stronger** than language {~}.

And likewise if two languages pick out the same truth tables, those languages are **expressively equivalent**. The languages {~, ∧, ∨} and {~, ∧}, for instance, are expressively equivalent: since both are expressively adequate, both pick out all possible truth tables.

A striking result about duality and expressive power is the following.

> A language is **expressively adequate** if (and only if) its **dual** is also **expressively adequate**.

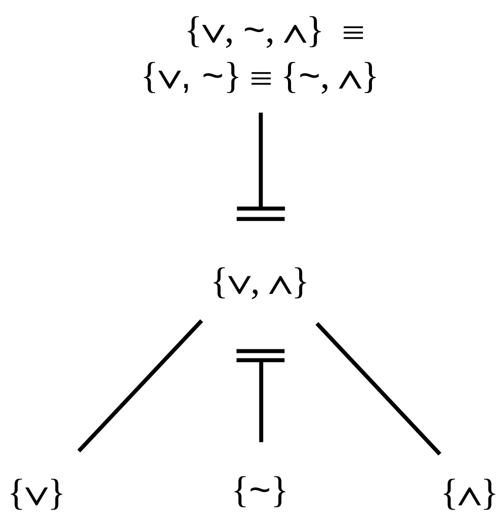
To pick one (rather uninteresting) example: {~, ∧, ∨} is expressively adequate, so its dual is expressively adequate. The example is uninteresting only because the language is a self-dual: "∧" is the dual of "∨," and "~" is its own dual.

A more interesting example: {~, ∧} is expressively adequate, so its dual, {~, ∨} is also expressively adequate.

That works for inadequacy as well: {∧} is expressively inadequate, and so is its dual {∨}. Likewise {~} is expressively inadequate and a self-dual; so (trivially) its dual is also expressively inadequate.

Appreciating this point about duality **cuts in half** the work necessary to prove expressive adequacy or inadequacy. For example: once we’ve established that language {~, ∧} is expressively adequate, we don’t need a *separate* proof that language {~, ∨} is. For {~, ∨} is the dual language of {~, ∧}, and so is bound to be adequate. Likewise, proving the inadequacy of {∧} suffices as well to prove the inadequacy of {∨}.[1]

We emphasize the connection between expressive power and duality in the following diagram. Here each set finds its dual on the other side of the center line – the left and right sides of the diagram are, in effect, logical ‘mirror images’. (The self-duals {∧, ~, ∨}, {∧, ∨}, and {~} thus straddle this line.) The three (equivalent) languages at the top are all expressively adequate.



These striking features of languages and expressive power stem from a point recognized in our previous discussion of duality.

Note that while we profile each formal language in terms of its set of connectives, a formal language is, more precisely, a set of sentences: just those sentences built

[1] Since {~} and {∧, ∨} are self-duals, proving their inadequacy saves us no work. But in general: whenever a language is not a self-dual, the proof of its adequacy or inadequacy spares us a proof for its dual.

from sentence letters and those connectives.[2] Each such set of sentences will likewise have a **dual set of sentences** (via Connective Swap): the set containing the dual of each sentence in the original set. And a set of truth tables will likewise have a **dual set of truth tables** (via True/False Swap).

Those two sets are bound to in parallel, thanks to an earlier parallel noted between Connective Swap and True/False Swap: a given formal sentence has a corresponding truth table (by the semantic rules), and the (Connective Swap) dual of that sentence will take the (True/False Swap) dual of that truth table.[3] Since a formal language is a set of sentences, which take a set of truth tables according to the semantic rules, we conclude that the (Connective Swap) dual set of sentences will take the (True/False Swap) dual set of truth tables.

> If a language picks out a certain set of truth tables, then its dual language will pick out the dual of those truth tables.

For instance, the set of sentences making up the language {∧, ~} will take a (Connective Swap) dual set of sentences making up the dual language {∨, ~}. And the set of truth tables picked out by the sentences of {∧, ~} will take as their (True/False Swap) dual the set of truth tables picked out by the sentences of {∨, ~}.

With that point in mind, it's easy to see that **a formal language is expressively inadequate if (and only if) its dual language is expressively inadequate**. If the formal language is inadequate, there is some truth table picked out by none of its sentences. But then the dual of that neglected truth table is likewise out of reach of the dual language – making that dual language inadequate as well

So likewise: **if a formal language is expressively adequate** – covering all of the truth tables with its sentences – **its dual language must also be expressively adequate** (and, of course, vice versa).

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[2] Since we take for granted that every formal language has sentence letters, various formal languages will differ only in terms of which connectives they have. That's why, even though a language is a set of sentences, we can pick out a specific formal language just in terms of its set of connectives.

[3] As noted in "*3.33. Logical Duality*," Section 2.